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The convergence of Yang-Mills integrals

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Abstract: We prove that SU(N) bosonic Yang-Mills matrix integrals are convergent for dimension (number of matrices) $D \geq D_c$. It is already known that $D_c = 5$ for $N = 2$; we prove that $D_c = 4$ for $N = 3$ and that $D_c = 3$ for $N \geq 4$. These results are consistent with the numerical evaluations of the integrals by Krauth and Staudacher.

Keywords: Non-perturbative Effects, M(atrix) Theories, Matrix Models.

JHEP02(2001)028

Contents

1. Introduction **1**

2. Preliminaries **3**

3. SU(2) **5**

4. SU(3) **5**

5. $N > 3$ **7**

6. Discussion **10**
- 6.1 Correlation functions 10
- 6.2 Supersymmetric integrals 11
- 6.3 Other Gauge groups 12

A. Divergent matrix integrals **12**

JHEP02(2001)028

1. Introduction

The discovery of D-branes and the realization of their importance in string theory and M-theory has led to a number of exciting conjectures relating these very complicated theories to (at least technically) much simpler M(atrix) theories [1] and the IKKT model for the type-IIB superstring [2] (for reviews see [3, 4]). This has generated renewed interest in supersymmetric Yang-Mills quantum mechanics [5]–[9] which is obtained by the the dimensional reduction of $D = 10$ supersymmetric Yang-Mills gauge theory (SSYM) to 1 remaining space-time dimension. Further dimensional reduction to 0 dimensions leads to the supersymmetric Yang-Mills matrix integrals which are an important component in calculating the Witten index for the quantum mechanics. Both the quantum mechanics and the matrix integrals exist for any dimension D in which the original SSYM exists, ie $D = 3$, 4, 6, and 10. Dropping the supersymmetry requirement leads to the bosonic Yang-Mills matrix integrals which exist for any D and are the main subject of this paper.

A couple of years ago Moore, Nekrasov and Shatashvili [10] found a way of calculating the partition function by deforming the Yang-Mills matrix integrals to a cohomological theory. Their method does not allow the calculation of arbitrary correlation functions in the original Yang-Mills picture where it is not known how to do exact calculations unless the gauge group is SU(2) [11]. However it is possible to do numerical calculations provided the gauge group is not too big. The numerically calculated partition functions for small $N = 3$, 4, and 5 [12, 13] agree with the cohomological calculations in [10]. Some correlation functions and eigenvalue distributions have also been obtained [13, 14]; in the case of $D = 4$ where the fermions can be integrated numerically by Monte Carlo many correlation functions have been found for values of N up to 48 [15]. In the course of their work Krauth and Staudacher [13] also investigated the properties of purely bosonic Yang-Mills integrals (these are defined below). It had been believed that the flat directions in the action would cause these integrals to diverge (their supersymmetric cousins being saved by the Pfaffian arising from the integration of the fermions which vanishes along the flat directions). Simple analytic calculations of the partition functions in the case of SU(2), and delicate numerical computations for SU(3), SU(4) and SU(5) (and subsequently for N up to 256 [16] and other gauge groups as well [17]) showed that this is not necessarily the case. Unfortunately up to now an analytic demonstration of the convergence of these integrals for SU($N > 2$) has been lacking. Our purpose here is to provide such a demonstration.

The bosonic Yang-Mills partition function for gauge group G in D "space-time" dimensions[1] dimensionally reduced to 0 is given by

$$\mathcal{Z}_{D,G} = \prod_{\mu=1}^{D} \int_{-\infty}^{\infty} dX_\mu \exp\left(\sum_{\mu>\nu} \mathrm{Tr}\,[X_\mu, X_\nu]^2\right), \tag{1.1}$$

where the matrices $\{X_\mu, \mu = 1, \dots, D\}$, which are traceless and hermitean, take values in the Lie algebra of G and can be written

$$X_\mu = \sum_{a=1}^{g} X_\mu^a t^a . \tag{1.2}$$

The $\{t^a, a = 1, \dots, g\}$ are the generators in the fundamental representation satisfying

$$\mathrm{Tr}\, t^a t^b = 2\delta^{ab} \tag{1.3}$$

and we shall use l to denote the rank of the Lie algebra. In this paper we will restrict ourselves to the groups SU(N). The measure and the integrand in (1.1) are then invariant under the SU(N) gauge symmetry

$$X_\mu \longrightarrow U^\dagger X_\mu U\,, \qquad U \in \mathrm{SU}(N) \tag{1.4}$$

[1] In the bosonic case there is no requirement of supersymmetry to restrict D so it is possible to consider D as a continuous variable by analytic continuation; however in this paper it is to be taken strictly as an integer.

JHEP02(2001)028

and the SO(D) symmetry

$$X_\mu \longrightarrow \sum_\nu Q_{\mu\nu} X_\nu \,, \qquad Q \in \mathrm{SO}(D)\,. \tag{1.5}$$

We will prove that $\mathcal{Z}_{D,\mathrm{SU}(N)}$ is convergent for dimension $D \geq D_c$ and divergent for $D < D_c$. It is already known from exact calculation that $D_c = 5$ for $N = 2$ (although we will show that our methods reproduce this almost trivially). We prove that $D_c = 4$ for $N = 3$ and that $D_c = 3$ for $N \geq 4$.

The body of this paper is concerned with establishing which integrals *converge*. In section 2 we set up our procedure and establish some results that are useful in every case. Section 3 deals with SU(2), section 4 with SU(3), and section 5 with SU($N > 3$). In appendix A we show which integrals *diverge*. We conclude with a brief discussion of the implications of our results for the supersymmetric theories and other gauge groups.

JHEP02(2001)028

2. Preliminaries

The dangerous regions which might cause the integral (1.1) to diverge are where all the commutators almost vanish but the magnitude of X_μ goes to infinity. Hence we let

$$X_\mu = R x_\mu \,, \qquad \mathrm{Tr}\, x_\mu x_\mu = 1\,, \tag{2.1}$$

where, as from now on, we use the summation convention for Greek indices. Then we have

$$\mathcal{Z}_{D,G} = \int_0^\infty R^{Dg-1} \mathcal{X}_{D,G}(R) dR\,, \tag{2.2}$$

where

$$\mathcal{X}_{D,G}(R) = \prod_{\nu=1}^{D} \int dx_\nu \;\, \delta(1 - \mathrm{Tr}\, x_\mu x_\mu) \exp\left(-R^4 S\right) \tag{2.3}$$

and

$$\begin{aligned} S &= -\frac{1}{2} \mathrm{Tr}\, [x_\mu, x_\nu]\, [x_\mu, x_\nu] \\ &= \frac{1}{2} \sum_{i,j,\mu,\nu} \left| [x_\mu, x_\nu]_{i,j} \right|^2 . \end{aligned} \tag{2.4}$$

We note that for any finite R the integral $\mathcal{X}_{D,G}(R)$ is bounded by a constant (since every term in the argument of the exponential is negative semi-definite) and therefore if for large R

$$\mathcal{X}_{D,G}(R) < \frac{\mathrm{const}}{R^\alpha}\,, \qquad \text{where} \qquad \alpha > Dg\,, \tag{2.5}$$

then the partition function $\mathcal{Z}_{D,G}$ is finite. Our tactic for proving convergence of $\mathcal{Z}_{D,G}$ is therefore to find a bound of the form (2.5) on $\mathcal{X}_{D,G}(R)$.

Now we split the integration region in (2.3) into two

$$\begin{aligned} \mathcal{R}_1 &: S < \left(R^{-(2-\eta)}\right)^2 , \\ \mathcal{R}_2 &: S \geq \left(R^{-(2-\eta)}\right)^2 , \end{aligned} \tag{2.6}$$

where η is small but positive. We see immediately that the contribution to $\mathcal{I}_{D,G}(R)$ from $\mathcal{R}_2$ is bounded by $A_1 \exp(-R^{2\eta})$ (we will use the capital letters A, B and C to denote constants throughout this paper) and thus automatically satisfies (2.5). Thus we can confine our efforts to the contribution from $\mathcal{R}_1$ in which we replace the exponential function by unity to get the bound

$$\mathcal{X}_{D,G}(R) < A_1 \exp(-R^{2\eta}) + \mathcal{I}_{D,G}(R) , \tag{2.7}$$

where

$$\mathcal{I}_{D,G}(R) = \prod_{\nu=1}^{D} \int_{\mathcal{R}_1} dx_\nu \, \delta(1 - \operatorname{Tr} x_\mu x_\mu) . \tag{2.8}$$

Since S is a sum of squares it follows that the region $\mathcal{R}_1'$ defined by

$$\left| [x_\mu, x_\nu]_{ij} \right| < R^{-(2-\eta)} , \qquad \forall \mu, \nu, i, j \tag{2.9}$$

is larger than the region $\mathcal{R}_1$ which we can therefore replace in (2.7) by $\mathcal{R}_1'$.

Now we utilise the SU(N) symmetry to diagonalise x_1 which we may therefore write as $x_1 =$ diag$(\lambda_1, \ldots, \lambda_N)$. The constraint (2.9) then becomes

$$|(\lambda_i - \lambda_j)(x_\nu)_{ij}| < R^{-(2-\eta)} . \tag{2.10}$$

This immediately leads to the generic case which is when the eigenvalues of x_1 are not degenerate

$$|\lambda_i - \lambda_j| > \epsilon , \tag{2.11}$$

where ϵ is a constant which we may choose but will always be finite. Then all the off-diagonal elements of $\{x_\nu, \nu = 2, \ldots, D\}$ are bounded by (2.10) and $\mathcal{I}_{D,G}(R)$ (2.8) has a contribution $\mathcal{I}^{\text{gen}}_{D,G}(R)$ which is bounded

$$\mathcal{I}^{\text{gen}}_{D,G}(R) < A_2 R^{-(2-\eta)(D-1)(g-l)} , \tag{2.12}$$

where the constant A_2 comes from the integral over all the diagonal elements (and does of course depend on ϵ). Note that once the off-diagonal elements are bounded by (2.10) and (2.11) all commutators are constrained to be $O(R^{-(2-\eta)})$ although the coefficient may be more than 1. Enforcing (2.9) implies constraints on the diagonal elements which lower A_2 in (2.12) but does not affect the power of R.

We note that the case when all eigenvalues are widely separated corresponds to the "perturbation" expansion of [16]; using (2.5) and (2.12) leads to the criterion $N > D/(D-2)$ for convergence that is given in [16]. It is fortuitous that this gives the correct conditions for convergence of $\mathcal{Z}_{D,G}$ because as we shall see $\mathcal{I}^{\text{gen}}_{D,G}$ is not usually the most divergent contribution.

JHEP02(2001)028

3. SU(2)

In the case of SU(2) each matrix x_μ has eigenvalues λ_μ and $-\lambda_\mu$. However the constraint in (2.1) implies that

$$2\lambda_\mu\lambda_\mu = 1 \tag{3.1}$$

and it follows that there must always be one matrix with eigenvalues of magnitude at least $(2D)^{-1/2}$. We can choose this matrix to be x_1 and hence by taking $\epsilon < 2(2D)^{-1/2}$ we always have the generic case. Hence

$$\mathcal{X}_{D,\mathrm{SU}(2)}(R) < A_1 \exp(-R^{2\eta}) + A_2 R^{-2(2-\eta)(D-1)} \tag{3.2}$$

and it follows immediately that $\mathcal{Z}_{D,\mathrm{SU}(2)}$ is finite for $D \geq 5$. We show in appendix A that $\mathcal{Z}_{D,\mathrm{SU}(2)}$ is divergent for smaller D. Of course these results are well known because $\mathcal{Z}_{D,\mathrm{SU}(2)}$ can be calculated exactly.

JHEP02(2001)028

4. SU(3)

From the constraint in (2.1) it follows that there must always be one matrix with an eigenvalue of magnitude at least $(3D)^{-1/2}$; as before we choose this matrix to be x_1. When all the eigenvalues of x_1 are separated by at least ϵ (2.11) we get the generic contribution (2.12) to the integral. However now we have the new possibility that two of the eigenvalues of x_1 become degenerate (it is not possible for all three eigenvalues to become degenerate provided we choose $\epsilon < \frac{1}{2}(3D)^{-1/2}$) in which case the condition (2.11) does not apply and we have to proceed more carefully; $\mathcal{I}_{D,\mathrm{SU}(3)}(R)$ is made up of a piece where (2.11) applies to all eigenvalues plus a new piece $\mathcal{I}^{\mathrm{deg}}_{D,\mathrm{SU}(3)}(R)$ from the region of integration where (2.11) is not satisfied by all eigenvalues,

$$\mathcal{I}_{D,\mathrm{SU}(3)}(R) = \mathcal{I}^{\mathrm{gen}}_{D,\mathrm{SU}(3)}(R) + \mathcal{I}^{\mathrm{deg}}_{D,\mathrm{SU}(3)}(R)\,. \tag{4.1}$$

First we write

$$x_1 = \frac{\rho_1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} \tilde{x}_1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

where $\tilde{x}_1 = \mathrm{diag}\,(\xi, -\xi)$; we know that $\left|2\rho_1/\sqrt{3}\right| > (3D)^{-1/2}$ and, because we are just interested in the case of degenerate eigenvalues, $2|\xi| < \epsilon$. Because only two of the three eigenvalues of x_1 are degenerate the constraint (2.10) bounds some of the off-diagonal elements of $x_{2,\ldots,D}$ so we can write these matrices in the form

$$x_\nu = \frac{\rho_\nu}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} \tilde{x}_\nu & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} + \\ + \begin{pmatrix} 0 & 0 & O(R^{-(2-\eta)}) \\ 0 & 0 & O(R^{-(2-\eta)}) \\ O(R^{-(2-\eta)}) & O(R^{-(2-\eta)}) & 0 \end{pmatrix}, \tag{4.3}$$

where $\tilde{x}_\nu$ is a 2×2 traceless hermitean matrix (i.e. it lives in the su(2) sub-algebra of the su(3) algebra inhabited by x_ν), and by $O(R^{-(2-\eta)})$ we mean that the elements are bounded by (2.10). At this stage the off-diagonal elements in the third row and column of the $x_{2,\dots,D}$ are innocuous and can be integrated out to get

$$\begin{aligned}\mathcal{I}^{\rm deg}_{D,{\rm SU}(3)}(R) < B_1 R^{-(2-\eta)(4(D-1)-2)} &\left(\int_{|\rho_1|>(4D)^{-1/2}} \prod_{\mu=1}^{D} d\rho_\mu\right)\left(\int_{\widetilde{\mathcal{R}}} d\xi \prod_{\nu=2}^{D} d\tilde{x}_\nu\right)(2\xi)^2 \times \\ &\times \Omega_{{\rm SU}(2)}\left(\sqrt{3}\rho_1-\xi\right)^2\left(\sqrt{3}\rho_1+\xi\right)^2\theta\Big(1-({\rm Tr}\,\tilde{x}_\alpha\tilde{x}_\alpha+2\rho_\alpha\rho_\alpha)\Big)\times \\ &\times\,\theta\Big(({\rm Tr}\,\tilde{x}_\alpha\tilde{x}_\alpha+2\rho_\alpha\rho_\alpha)-\left(1-4\left(\epsilon R^{(2-\eta)}\right)^{-2}\right)\Big)\,, \end{aligned}\qquad (4.4)$$

where θ denotes the step function, we have included the Vandermonde determinant for x_1, and $\Omega_{{\rm SU}(2)}$ is the volume of SU(2). The region $\widetilde{\mathcal{R}}$ is defined by $|\xi|<\epsilon/2$ and

$$\left|[\tilde{x}_\mu,\tilde{x}_\nu]_{ij}\right| < R^{-(2-\eta)}\,,\qquad \forall\mu,\nu,i,j\,. \qquad (4.5)$$

Within the region of integration we can bound the factors $(3\rho_1\pm\xi)^2$ by a constant and then, because we are looking for an upper bound, we can drop the constraints $|\xi|<\epsilon/2$ and $|\rho_1|>(4D)^{-1/2}$. Doing the ρ_μ integrals we are left with

$$\mathcal{I}^{\rm deg}_{D,{\rm SU}(3)}(R) < B_2 R^{-(2-\eta)4(D-1)}\mathcal{F}_{D,{\rm SU}(2)}\,, \qquad (4.6)$$

where

$$\begin{aligned}\mathcal{F}_{D,{\rm SU}(2)} &= \left(\int_{\widetilde{\mathcal{R}}} d\xi\prod_{\nu=2}^{D} d\tilde{x}_\nu\right)(2\xi)^2\Omega_{{\rm SU}(2)}\,\theta(1-{\rm Tr}\,\tilde{x}_\mu\tilde{x}_\mu) \\ &= (2-\eta)R^{-(2-\eta)}\int_0^R du\,u^{1-\eta}\left(\int_{\widetilde{\mathcal{R}}}\prod_{\nu=1}^{D} d\tilde{x}_\nu\right)\times \\ &\qquad\qquad \times\,\delta\left(\left[\frac{u}{R}\right]^{2-\eta}-{\rm Tr}\,\tilde{x}_\mu\tilde{x}_\mu\right). \end{aligned}\qquad (4.7)$$

Making the rescaling $\tilde{x}_\mu=\tilde{y}_\mu[u/R]^{1-\eta/2}$ we find that

$$\mathcal{F}_{D,{\rm SU}(2)} = R^{-(2-\eta)3D/2}\int_0^R u^{3D(1-\eta/2)-1}\mathcal{I}_{D,{\rm SU}(2)}(u)\,du\,. \qquad (4.8)$$

As $R\to\infty$ we use the results from section 3 to bound the remaining integral giving

$$\begin{aligned}\mathcal{F}_{D,{\rm SU}(2)} < \;& B_3 R^{-(2-\eta)3D/2}\,, \qquad D\geq 5\,, \\ & B_4 R^{-(2-\eta)3D/2}\log R\,, \qquad D=4\,, \\ & B_5 R^{-(2-\eta)(3D-1)/2}\,, \qquad D=3 \end{aligned}\qquad (4.9)$$

JHEP02(2001)028

and so we find that

$$\mathcal{I}^{\rm deg}_{D,{\rm SU}(3)}(R) < B_6 \left(\frac{1}{R^{2-\eta}}\right)^{4(D-1)+(3D-\delta_{D,3})/2} (\log R)^{\delta_{D,4}} . \qquad (4.10)$$

From (2.12) we know that

$$\mathcal{I}^{\rm gen}_{D,{\rm SU}(3)}(R) < B_7 R^{-(2-\eta)6(D-1)} . \qquad (4.11)$$

Applying the criterion (2.5) we see that both $\mathcal{I}^{\rm deg}_{D,{\rm SU}(3)}(R)$ and $\mathcal{I}^{\rm gen}_{D,{\rm SU}(3)}(R)$ make a finite contribution to $\mathcal{Z}_{D,{\rm SU}(3)}$ only for $D \geq 4$. It is straightforward to check that $\mathcal{Z}_{D=3,{\rm SU}(3)}$ diverges (see appendix A). Thus we conclude that $\mathcal{Z}_{D,{\rm SU}(3)}$ is finite for $D \geq 4$ only.

5. $N > 3$

The argument for higher N has the same structure as for SU(3). Again there is a generic contribution which is bounded as shown in (2.12) and a degenerate contribution. As usual x_1 can be chosen so that at least one of its eigenvalues has magnitude $(ND)^{-1/2}$ or more. The difference is that x_1 can have not only two degenerate but $2, 3, \ldots, N-1$ and two or more sets of them. To deal with this we note that whenever x_1 has some configuration of exactly degenerate eigenvalues there is a sub-algebra of su(N) which commutes with x_1. (As we saw in section 4 for the case of SU(3) the only possibility is to have two degenerate eigenvalues and the sub-algebra is su(2).) Suppose that x_1 has K sets of degenerate eigenvalues with degeneracies $\{N_1, \ldots, N_K\}$; then the sub-algebra which commutes with x_1 is

$$H = \bigoplus_{k=1}^{K} {\rm su}(N_k) . \qquad (5.1)$$

Now let $\bar{t}^a$ denote those generators of G that do not lie in H, and decompose x_μ into

$$\begin{aligned} x_1 &= {\rm diag}\left(\rho_1^1 I^{N_1}, \ldots, \rho_1^K I^{N_K}, \sigma_1^1 \ldots, \sigma_1^M\right) + {\rm diag}\left(\tilde{x}_1^1, \ldots, \tilde{x}_1^K, 0 \ldots, 0\right), \\ x_{\mu>1} &= {\rm diag}\left(\rho_\mu^1 I^{N_1}, \ldots, \rho_\mu^K I^{N_K}, \sigma_\mu^1 \ldots, \sigma_\mu^M\right) + {\rm diag}\left(\tilde{x}_\mu^1, \ldots, \tilde{x}_\mu^K, 0 \ldots, 0\right) + \\ &\quad + \sum_a \tau_\mu^a \bar{t}^a . \end{aligned} \qquad (5.2)$$

Here I^N denotes the $N \times N$ Identity matrix, $\tilde{x}_\mu^k$ lies in the Lie algebra su(N_k) with $\tilde{x}_1^k$ diagonal, M is given by

$$M = N - \sum_{k=1}^{K} N_k \qquad (5.3)$$

JHEP02(2001)028

and the tracelessness condition is

$$0 = \sum_{k=1}^{M} \sigma_\mu^k + \sum_{k=1}^{K} N_k \rho_\mu^k \,. \tag{5.4}$$

Note that the ordering of ρs and σs in (5.2) is not significant; for example we could chose to take x_1 so that the elements are in decreasing order down the diagonal and then degenerate and non-degenerate eigenvalues would be all mixed up in general. If $\tilde{x}_1^{1,\ldots,K} = 0$ then the eigenvalues of x_1 are exactly degenerate; we have K blocks of eigenvalues $\rho_1^{1,\ldots,K}$ with degeneracy N_k and singleton eigenvalues $\sigma_1^{1,\ldots,M}$. When the exact degeneracy is relaxed slightly we have blocks of eigenvalues $\lambda_m^k, m = 1, \ldots, N_k$ with each block having central value

$$\frac{1}{N_k} \sum_{m=1}^{N_k} \lambda_m^k = \rho_1^k \,, \tag{5.5}$$

together with the singleton eigenvalues $\lambda_1^{(K+j)} = \sigma_1^j$, $j = 1, \ldots, M$.

For each sub-algebra H there is a contribution $\mathcal{I}^H_{D,\mathrm{SU}(N)}(R)$ to $\mathcal{I}^{\mathrm{deg}}_{D,\mathrm{SU}(N)}(R)$. This comes from the integration region $\mathcal{P}$ where the $\rho_1^k, \sigma_1^j, \tilde{x}_1^k$ are such that the eigenvalues of x_1 satisfy

$$\begin{aligned} \mathcal{P}: \ & \left|\lambda_m^k - \lambda_{m'}^{k'}\right| > \epsilon \,, \qquad k \neq k' \,, \qquad m = 1, \ldots, N_k \,, \qquad m' = 1, \ldots, N_{k'} \,, \\ & \left|\lambda_m^k - \lambda_{m+1}^k\right| \leq \epsilon \,, \qquad m = 1, \ldots, N_k - 1 \,. \end{aligned} \tag{5.6}$$

We note that any sequence of eigenvalues can be arranged in the manner implied by (5.6) for some H. Therefore by considering the $\mathcal{I}^H_{D,\mathrm{SU}(N)}(R)$ for all possible H we exhaust all possible nearly degenerate eigenvalue configurations for x_1. That is to say the degenerate term in $\mathcal{I}_{D,\mathrm{SU}(N)}(R)$ is now the sum of contributions from all the possible Hs

$$\mathcal{I}^{\mathrm{deg}}_{D,\mathrm{SU}(N)}(R) = \sum_H \mathcal{I}^H_{D,\mathrm{SU}(N)}(R) \,. \tag{5.7}$$

Now we bound the $\mathcal{I}^H_{D,\mathrm{SU}(N)}(R)$. The Vandermonde determinant for x_1 is easily bounded from above through

$$\begin{aligned} \Delta &= \prod_{k>k',m,m'} (\lambda_m^k - \lambda_{m'}^{k'})^2 \prod_{k,m>m'} (\lambda_m^k - \lambda_{m'}^k)^2 \\ &< 2^{n_H} \prod_{k,m>m'} (\lambda_m^k - \lambda_{m'}^k)^2 \,, \\ n_H &= N(N-1) - \sum_{k=1}^{K} N_k(N_k - 1) \,, \end{aligned} \tag{5.8}$$

where we have used the fact that none of the eigenvalues can have magnitude more than 1 on account of the constraint (2.1). Of course this bound is simply the product

JHEP02(2001)028

of the Vandermonde determinants for the constituent $\mathrm{su}(N_k)$ factors of the subalgebra H. Now we note that the τ^a_μ, which are those off-diagonal elements of $x_{2,\ldots,D}$ that do not lie in H, are constrained by (2.10) and, following our procedure in the SU(3) case, we integrate them out to get

$$\mathcal{I}^H_{D,\mathrm{SU}(N)}(R) < C_1 R^{-(2-\eta)((D-1)n_H-2)} \int_{\mathcal{P}} \prod_{\mu=1}^{D} \left(\prod_{k=1}^{K} d\rho^k_\mu \prod_{j=1}^{M-1} d\sigma^j_\mu \right) \times \tag{5.9}$$
$$\times \prod_{k=1}^{K} \left(\int_{\widetilde{\mathcal{R}}^k} \prod_{\nu=1}^{D} d\tilde{x}^k_\nu \right) \theta\left(1 - \sum_{k=1}^{K} \left(\mathrm{Tr}\, \tilde{x}^k_\mu \tilde{x}^k_\mu + N_k \rho^k_\mu \rho^k_\mu \right) - \sum_{j=1}^{M} \sigma^j_\mu \sigma^j_\mu \right) \times$$
$$\times\, \theta\left(\sum_{k=1}^{K} \mathrm{Tr}\, \tilde{x}^k_\mu \tilde{x}^k_\mu + N_k \rho^k_\mu \rho^k_\mu + \sum_{j=1}^{M} \sigma^j_\mu \sigma^j_\mu - \left(1 - n_H \left(\epsilon R^{(2-\eta)} \right)^{-2} \right) \right),$$

where σ^M_μ is given by (5.4), the region $\widetilde{\mathcal{R}^k}$ is defined by

$$\left| \left[\tilde{x}^k_\mu, \tilde{x}^k_\nu \right]_{ij} \right| < R^{-(2-\eta)}, \qquad \forall \mu, \nu, i, j\,. \tag{5.10}$$

The r.h.s. of (5.9) is now bounded above by dropping the $\mathcal{P}$ constraint and integrating out the ρ^k_μ and σ^j_μ which leaves us with

$$\begin{aligned}\mathcal{I}^H_{D,\mathrm{SU}(N)}(R) &< C_2 R^{-(2-\eta)(D-1)n_H} \prod_{k=1}^{K} \left(\int_{\widetilde{\mathcal{R}}^k} \prod_{\nu=1}^{D} d\tilde{x}^k_\nu \right) \theta\left(1 - \sum_{k=1}^{K} \mathrm{Tr}\, \tilde{x}^k_\mu \tilde{x}^k_\mu \right) \\ &< C_2 R^{-(2-\eta)(D-1)n_H} \prod_{k=1}^{K} \mathcal{F}_{D,\mathrm{SU}(N_k)}\,, \end{aligned} \tag{5.11}$$

where

$$\mathcal{F}_{D,\mathrm{SU}(N_k)} = \left(\int_{\widetilde{\mathcal{R}}^k} \prod_{\nu=1}^{D} d\tilde{x}^k_\nu \right) \theta\left(1 - \mathrm{Tr}\, \tilde{x}^k_\mu \tilde{x}^k_\mu \right). \tag{5.12}$$

We note in passing that if H is empty then (5.11) simply reduces to the generic case (2.12). We now repeat the steps (4.7) to (4.9) to find that

$$\mathcal{F}_{D,\mathrm{SU}(N_k)} = R^{-(2-\eta)D(N_k^2-1)/2} \int_0^R u^{D(N_k^2-1)(1-\eta/2)-1} \mathcal{I}_{D,\mathrm{SU}(N_k)}(u)\, du\,. \tag{5.13}$$

The final step is by induction on N:

1. $\mathcal{F}_{D,\mathrm{SU}(2)}$ is given in (4.9).

2. From our results for SU(3) (4.10) and (4.11) we deduce that

$$\begin{aligned}\mathcal{F}_{D,\mathrm{SU}(3)} < {} & C_3 R^{-4(2-\eta)D}\,, \qquad D \geq 4\,, \\ & C_4 R^{-4(2-\eta)D} \log R\,, \qquad D = 3\,. \end{aligned} \tag{5.14}$$

JHEP02(2001)028

3. For gauge group SU(4) the possible sub-algebras, H, are su(2), su(2) $\oplus$ su(2) and su(3). For $D = 3$ we find that

$$\mathcal{I}_{3,\mathrm{SU}(4)}(R) < C_5 R^{-48} \log R \tag{5.15}$$

and hence the integral for $\mathcal{Z}_{3,\mathrm{SU}(4)}$ converges. For $D \geq 4$ it is simple to check that $\mathcal{Z}_{D,\mathrm{SU}(4)}$ converges and the dominant term in $\mathcal{I}_{D,\mathrm{SU}(4)}(R)$ comes from $H =$ su(3) (we will give a general formula for the behaviour of $\mathcal{I}_{D,\mathrm{SU}(N)}(R)$ below).

4. We now assume that convergence is established for $D \geq 3$ for $N < N^*$. As a consequence we have that

$$\mathcal{F}_{D,\mathrm{SU}(N)} < CR^{-(2-\eta)D(N^2-1)/2}\,, \qquad D \geq 3 \qquad \text{and} \qquad 3 < N < N^*\,, \tag{5.16}$$

together with the slightly different bounds (4.9) and (5.14) for $N = 2$ and 3. This is enough information to bound $\mathcal{I}^H_{D,\mathrm{SU}(N^*)}(R)$ using (5.11) for any sub-algebra H

$$\begin{aligned}\mathcal{I}^H_{D,\mathrm{SU}(N^*)}(R) &< C_6 R^{-(2-\eta)((D-1)n^*_H + \frac{1}{2}D\sum_k (N_k^2-1))} \times \\ &\quad\times \left[R^{n_2\delta_{D,3}} (\log R)^{n_2\delta_{D,3}+n_3\delta_{D,4}}\right],\end{aligned} \tag{5.17}$$

where $n_{2,3}$ denotes the number of su(2) and su(3) factors, respectively in H. It is straightforward to check that for $D \geq 3$ the slowest decay at large R occurs when $H = \mathrm{su}(N-1)$; basically this minimises the number of R^{-2} factors coming from off-diagonal elements not in the sub-algebra and maximises the number of R^{-1} factors coming from elements in the sub-algebra. Thus for $D \geq 3$ and $N \geq 4$ we have

$$\mathcal{I}^{\mathrm{deg}}_{D,\mathrm{SU}(N)}(R) < CR^{-(4(D-1)(N-1)+DN(N-2))} (\log R)^{\delta_{N,4}\delta_{D,3}}\,. \tag{5.18}$$

Applying the criterion (2.5) shows that both generic (2.12) and degenerate terms make a finite contribution to $\mathcal{Z}_{D,\mathrm{SU}(N)}$ for $D \geq 3$, $N \geq 4$.

This completes our proof.

6. Discussion

6.1 Correlation functions

We can extend the definition of the partition function (1.1) to correlation functions so that

$$\langle\,\cdot\,\rangle = \prod_{\mu=1}^{D} \int_{-\infty}^{\infty} dX_\mu (\cdot) \exp\left(\sum_{\mu>\nu} \mathrm{Tr}\,[X_\mu, X_\nu]^2\right), \tag{6.1}$$

JHEP02(2001)028

where $(\cdot)$ represents some kind of product of the X_μ with P factors. Making the change of variables (2.1), realising that the absolute value of the corresponding product of the x_μ must be bounded by a constant, and using (5.18) we get that

$$|\langle \cdot \rangle| < C_8 \int_0^R dR\, R^{-2N(D-2)+3D-5}\, R^P\,. \tag{6.2}$$

Thus correlators with fewer than

$$P_c = 2N(D-2) - 3D + 4 \tag{6.3}$$

factors are guaranteed to be finite; of course correlators with more factors than this *may* be finite but then they must have some special property so that the leading divergences cancel. The authors of [14] "guessed" on the basis of reasonable arguments that the eigenvalue density $\rho(\lambda)$ for X_μ behaves like

$$\rho(\lambda) \sim \text{const}\, \lambda^{-2N(D-2)+3D-5} \tag{6.4}$$

at large λ. This is completely consistent with our results.

6.2 Supersymmetric integrals

The supersymmetric partition functions are given by

$$\mathcal{Z}^{SS}_{D,G} = \prod_{\mu=1}^{D} \int_{-\infty}^{\infty} dX_\mu\, \mathcal{P}_{D,G}(X_\mu)\, \exp\left(\sum_{\mu>\nu} \mathrm{Tr}\, [X_\mu, X_\nu]^2 \right), \tag{6.5}$$

where the Pfaffian $\mathcal{P}_{D,G}$ arises from integrating out the fermionic degrees of freedom and is a homogeneous polynomial of degree $(D-2)(N^2-1)$. We can of course regard $\mathcal{Z}^{SS}_{D,G}$ as being a correlation function in the bosonic theory and apply the considerations of section 6.1 to it for $P = (D-2)(N^2-1)$; we find immediately that all the supersymmetric partition functions are naively divergent. However the Pfaffian contains many terms with different signs and we expect many cancellations. The simplest example is for SU(2) where the Pfaffians are known explicitly [12] and (except for $D = 3$ where $\mathcal{Z}^{SS}_{3,\mathrm{SU}(2)} = 0$ because $\mathcal{P}_{3,\mathrm{SU}(2)}$ is an odd function) can be expressed as sums of powers of commutators $[X_\mu, X_\nu]$. This is particularly convenient with our method because the rescaling in (2.1) followed by the restriction to the region $\mathcal{R}_1$ in (2.6) means that we can bound $\mathcal{Z}^{SS}_{D,\mathrm{SU}(2)}$ simply by setting all commutators to a constant. It follows that if the bosonic partition function converges so does the supersymmetric one; thus the $D = 6$ and 10 partition functions are convergent but $D = 4$ is marginal and we would need to work harder (in fact it is known to converge). The situation with bigger N is more complicated mainly because relatively little is known about the Pfaffians and we will return to this problem in a separate paper.

JHEP02(2001)028

6.3 Other Gauge groups

In this paper we have concentrated on SU(N) gauge groups. However all the results we obtain depend in a well-defined way on simple group theoretical properties such as the order, rank and sub-algebras of Lie algebras. It is therefore tempting to suppose that expressed in this form our results would carry over directly to any Lie group. However some of the steps we have made such as the diagonalization of x_1 and the inductive argument in section 5 do depend on the group being SU(N) and we will deal with the other groups in a separate paper.

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A. Divergent matrix integrals

The $D = 2$ integral

$$\mathcal{Z}_{2,G} = \int_{-\infty}^{\infty} dX_1 dX_2 \exp\left(\mathrm{Tr}\,[X_1, X_2]^2\right) \tag{A.1}$$

is divergent for all SU(N). This is easily seen by diagonalizing X_1; the integrand then does not depend upon the diagonal elements of X_2 and so the integral over them diverges.

Some other low N and low D integrals are divergent. To see this we go back to (1.1), diagonalize X_1, and separate out the diagonal elements for $\nu > 1$,

$$X_\nu = \mathrm{diag}(\lambda_{\nu 1}, \ldots, \lambda_{\nu N}) + X_\nu^\perp\,. \tag{A.2}$$

We then change variables from the $X_\nu^\perp$ to $(D-1)(g-l)$ dimensional polar coordinates with radial variable r and angular variables $\{\theta_i\}$. The integral over all the diagonal elements is gaussian so we do it and are left with

$$\mathcal{Z}_{D,G} = \int_0^\infty r^{D(g-2l)-2g+2l-1} dr \int d\Omega F_1(\{\theta_i\}) \exp\left(-r^4 F_2(\{\theta_i\})\right), \tag{A.3}$$

where F_1 and F_2 are horrible but positive semi-definite functions and Ω is the $(D-1)(g-l)$ dimensional solid angle. We see immediately that the integral over r diverges at $r = 0$ if

$$D \leq \frac{2(g-l)}{g-2l} \tag{A.4}$$

so we deduce that the $D = 3, 4$ integrals diverge for SU(2) and that the $D = 3$ integral diverges for SU(3).

JHEP02(2001)028

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JHEP02(2001)028

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JHEP02(2001)028